

EXACT SOLUTIONS OF NONCOMMUTATIVE VACUUM EINSTEIN FIELD EQUATIONS AND PLANE-FRONTED GRAVITATIONAL WAVES

DING WANG, R. B. ZHANG, AND XIAO ZHANG

ABSTRACT. We construct a class of exact solutions of the noncommutative Einstein field equations in the vacuum, which are noncommutative analogues of the plane-fronted gravitational waves in classical gravity.

1. INTRODUCTION

There have been intensive research activities on noncommutative relativity in recent years. In particular, several tentative proposals [1, 2, 3, 4, 5, 6, 7] for a theory of noncommutative relativity have been put forward. In [2] noncommutativity was introduced into gravity by deforming the diffeomorphism algebra. In [1, 5] general relativity on a noncommutative spacetime is treated as a noncommutative gauge theory. Very recently, the authors of [8] explored a possible moving frame formalism for a noncommutative geometry on the Moyal space as the first step toward setting up a framework for noncommutative general relativity. Much work has also been done to investigate implications of spacetime noncommutativity to black hole physics [9, 10, 5].

In the papers [6, 7], a formalism for spacetime quantisation was proposed, which made use of isometric embeddings [11, 12, 13] of spacetime into pseudo-Euclidean spaces. In this formalism, one first finds a global embedding of a spacetime into some pseudo-Euclidean space, whose existence is guaranteed by theorems of Nash, Clarke and Greene [11, 12, 13]. Then one quantises the spacetime following the strategy of deformation quantisation [14, 15] by deforming [16] the algebra of functions in the pseudo-Euclidean space to a noncommutative associate algebra known as the Moyal algebra. Through this mechanism, classical spacetime metrics will deform to “quantum” noncommutative metrics which acquire quantum fluctuations. In particular, certain anti-symmetric components arise in the deformed metrics, which involve the Planck constant and vanish in the classical limit.

The theory of [6, 7, 9] can be formulated in an intrinsic way, free of the use of embeddings. This theory retains the notions of connections and curvatures in the noncommutative setting in a mathematically consistent manner. In particular, the quantum deformed noncommutative Ricci curvatures can be defined in a unique way. This enabled one to develop a noncommutative analogue of the Einstein field equations [6, 7].

It is important to solve the noncommutative Einstein field equations to construct quantum noncommutative spacetimes. In [9], we obtained noncommutative analogues of Schwarzschild spacetime and de-Sitter Schwarzschild spacetime,

which are approximate solutions of the noncommutative Einstein field equations exact to the first order of the deformation parameter. Quantum corrections to the area law of black hole entropy was observed for these solutions.

In this letter we construct a class of *exact* solutions of the noncommutative Einstein field equations in the vacuum. These solutions are quantum deformations of the plane-fronted gravitational waves first constructed by Brinkmann in 1925 [17] and have since been rediscovered several times (e.g. [18, 19, 20]). Our solutions are noncommutative gravitational analogues of electromagnetic plane waves. We expect them to have an important role to play in future investigations of quantum gravity.

Fuzzy pp-waves were constructed by Madore, Maceda and Robinson in [21]. These authors start with a given classical solution of the Einstein field equation in the vacuum and construct a noncommutative algebra and a differential calculus which supported the metric. The corresponding noncommutative scalar curvature was however nonzero. In general the quantum deformed metrics of most classical spacetimes satisfy the field equations only up to some order of the Planck constant.

Exact (that is, not approximate) solutions of noncommutative Einstein field equations do not seem to have been investigated much in the literature. Presumably this is partly due to the fact that many of the proposals of noncommutative relativity are based on intuition. Much work has been done to investigate corrections to physically relevant quantities to the first order in the deformation parameter within the frameworks of the various proposals. However, to go beyond the first order approximation, one will need a mathematically more rigorous theory. In particular, we can only investigate exact solutions when precisely formulated noncommutative Einstein field equations are given.

We mention that even within the mathematically rigorous formulations like those of [21] and [6, 7], the mathematical complexities introduced by spacetime noncommutativity makes it extremely difficult to study exact solutions of the noncommutative Einstein field equations. Thus it is quite remarkable that the quantum deformed plane-fronted gravitational waves constructed here solve the noncommutative vacuum Einstein field equations exactly.

2. NONCOMMUTATIVE EINSTEIN EQUATIONS

In order to set up the noncommutative Einstein equations, we need to have a noncommutative differential geometry which retains the notions of metric, connection and curvature. Such a theory was constructed in [6, 7]. We describe the theory very briefly here; details can be found in [6, 7].

2.1. A local noncommutative differential geometry. Let U be a domain in \mathbb{R}^n with natural coordinates $\{x^0, \dots, x^n\}$. Let \bar{h} be a real indeterminate, and denote by $\mathbb{R}[[\bar{h}]]$ the ring of formal power series in \bar{h} . Let \mathcal{A} be the set of formal power series in \bar{h} with coefficients being real smooth functions on U . Namely, every element of \mathcal{A} is of the form $\sum_{i \geq 0} f_i \bar{h}^i$ where f_i are smooth functions on U . Then \mathcal{A} is an $\mathbb{R}[[\bar{h}]]$ -module.

Given any two smooth functions u and v on U , we denote by uv the usual point-wise product of the two functions. We also define their star-product (or

more precisely, Moyal product) $u * v$ on U by

$$(2.1) \quad (u * v)(x) = \lim_{x' \rightarrow x} \exp \left(\bar{h} \sum_{ij} \theta_{ij} \partial_i \partial'_j \right) u(x) v(x'),$$

where $\partial_i = \frac{\partial}{\partial x^i}$, and (θ_{ij}) is a constant skew symmetric $n \times n$ matrix. It is well known that such a multiplication is associative. Thus \mathcal{A} equipped with the Moyal product is a deformation of the algebra of functions on U in the sense of [16]. Since θ is constant, the Leibniz rule remains valid in the present case:

$$\partial_i(u * v) = \partial_i u * v + u * \partial_i v.$$

In noncommutative geometry [24], the associative algebra \mathcal{A} is regarded as defining some *quantum deformation of the region U* , and finitely generated projective modules over \mathcal{A} are regarded as (spaces of sections of) noncommutative vector bundles on the quantum deformation of U (defined by the noncommutative algebra \mathcal{A}). Given an integer $m > n$, we let ${}_l\mathcal{A}^m$ (resp. \mathcal{A}_r^m) be the set of m -tuples with entries in \mathcal{A} written as rows (resp. columns). We shall regard ${}_l\mathcal{A}^m$ (resp. \mathcal{A}_r^m) as a left (resp. right) \mathcal{A} -module with the action defined by multiplication from the left (resp. right). More explicitly, for $v = (a_1 \ a_2 \ \dots \ a_m) \in {}_l\mathcal{A}^m$, and $b \in \mathcal{A}$, we have $b * v = (b * a_1 \ b * a_2 \ \dots \ b * a_m)$. Similarly for

$$w = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \in \mathcal{A}_r^m, \text{ we have } w * b = \begin{pmatrix} a_1 * b \\ a_2 * b \\ \vdots \\ a_m * b \end{pmatrix}. \text{ Let } \mathbf{M}_m(\mathcal{A}) \text{ be the set of } m \times m\text{-}$$

matrices with entries in \mathcal{A} . We define matrix multiplication in the usual way but by using the Moyal product for products of matrix entries, and still denote the corresponding matrix multiplication by $*$. Now for $A = (a_{ij})$ and $B = (b_{ij})$, we have $(A * B) = (c_{ij})$ with $c_{ij} = \sum_k a_{ik} * b_{kj}$. Then $\mathbf{M}_m(\mathcal{A})$ is an $\mathbb{R}[[\bar{h}]]$ -algebra, which has a natural left (resp. right) action on \mathcal{A}_r^m (resp. ${}_l\mathcal{A}^m$).

A finitely generated projective left (reps. right) \mathcal{A} -module is isomorphic to some direct summand of ${}_l\mathcal{A}^m$ (resp. \mathcal{A}_r^m) for some $m < \infty$. If $e \in \mathbf{M}_m(\mathcal{A})$ satisfies the condition $e * e = e$, that is, it is an idempotent, then

$$\mathcal{M} = {}_l\mathcal{A}^m * e := \{v * e \mid v \in {}_l\mathcal{A}^m\}, \quad \tilde{\mathcal{M}} = e * \mathcal{A}_r^m := \{e * w \mid w \in \mathcal{A}_r^m\}$$

are respectively projective left and right \mathcal{A} -modules. Furthermore, every projective left (right) \mathcal{A} -module is isomorphic to an \mathcal{M} (resp. $\tilde{\mathcal{M}}$) constructed this way by using some idempotent e .

As the noncommutative geometries on the left module \mathcal{M} and right module $\tilde{\mathcal{M}}$ are equivalent, we need only to investigate \mathcal{M} . Let

$$\omega_i = -\partial_i e$$

be the canonical connections on \mathcal{M} . The covariant derivative on the noncommutative bundle \mathcal{M} is given by

$$\nabla_i \zeta = \partial_i \zeta + \zeta * \omega_i, \quad \forall \zeta \in \mathcal{M}.$$

The *curvature* of \mathcal{M} associated with the connection ω_i is given by

$$\mathcal{R}_{ij} = \partial_i \omega_j - \partial_j \omega_i - [\omega_i, \omega_j]_*,$$

where $[\omega_i, \omega_j]_* = \omega_i * \omega_j - \omega_j * \omega_i$ is the commutator.

Let $\eta = \text{diag}(1, \dots, 1, -1, \dots, -1)$ be a diagonal $m \times m$ matrix with p of the diagonal entries being 1, and $q = m - p$ of them being -1 for some p . The *fibre metric* is the \mathcal{A} -bilinear map

$$(2.2) \quad \mathbf{g} : \mathcal{M} \otimes_{\mathbb{R}[[\hbar]]} \tilde{\mathcal{M}} \longrightarrow \mathcal{A}, \quad v \otimes w \mapsto v * w,$$

where for any $v = (v_1 \ \dots \ v_m) \in \mathcal{M}$ and $w = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} \in \tilde{\mathcal{M}}$, we have $v * w =$

$\sum_{i=1}^m v_i * w_i$. The metric compatibility of the connection and also the Bianchi identities for the Riemannian curvature were discussed in [6].

In certain situations, we may regard \mathcal{M} and $\tilde{\mathcal{M}}$ as noncommutative tangent bundles of some noncommutative space. Consider the case when there exists a finite set of \mathcal{A} -generators E_i ($i = 1, 2, \dots, n$) of \mathcal{M} with the following properties. The column vectors $\eta(E_i)^t$ (where $(E_i)^t$ are the transposes of E_i) generate $\tilde{\mathcal{M}}$, and the $n \times n$ matrix (\mathbf{g}_{ij}) with

$$(2.3) \quad \mathbf{g}_{ij} = \mathbf{g}(E_i, \eta(E_j)^t)$$

is invertible over U . Then the idempotent e is given by $e = \eta(E_i)^t * \mathbf{g}^{ij} * E_j$. In this case, we call the matrix $\mathbf{g} := (\mathbf{g}_{ij})$ the *metric*.

We may consider the components of the curvature \mathcal{R}_{ij} :

$$R_{kij}^l = E_k * \mathcal{R}_{ij} * \tilde{E}^l, \quad R_{lkij} = R_{kij}^p * \mathbf{g}_{pl},$$

where $\tilde{E}^i = \eta(E_j)^t * \mathbf{g}^{ji}$, $E^i = \mathbf{g}^{ij} * E_j$, for $i = 1, 2, \dots, n$.

A new feature is that there are two consistent ways to contract \mathcal{R}_{ij} , leading to two distinct noncommutative Ricci curvatures R_j^i and Θ_j^i respectively defined by

$$R_j^i = E^i * \mathcal{R}_{pj} * \tilde{E}^p, \quad \Theta_j^i = E^p * \mathcal{R}_{jp} * \tilde{E}^i.$$

Correspondingly there are two scalar curvatures $R = R_i^i$ and $\Theta = \Theta_i^i$. Both R_j^i and Θ_j^i reduce to the usual Ricci curvature in the commutative limit.

We now state the *noncommutative Einstein field equations in the vacuum* (for unknowns E_i) in this theory, which are given by

$$(2.4) \quad R_j^i = 0, \quad \Theta_j^i = 0, \quad \text{for all } i, j.$$

The aim of this note is to construct exact solutions of the equations.

2.2. Embedded noncommutative spaces. Embedded noncommutative spaces are elementary and manifestly consistent realisations of the local differential geometry discussed above. Given $X = (X^1 \ X^2 \ \dots \ X^m)$ in ${}_{\iota}\mathcal{A}^m$, we define an $n \times n$ matrix $(\mathbf{g}_{ij})_{i,j=1,2,\dots,n}$ with entries

$$(2.5) \quad \mathbf{g}_{ij} = \sum_{\alpha=1}^m \partial_i X^\alpha * \eta_{\alpha\beta} * \partial_j X^\beta,$$

where $\eta_{\alpha\beta} = \pm \delta_{\alpha\beta}$ are the matrix elements of the diagonal matrix η .

The matrix $\mathbf{g} = (\mathbf{g}_{ij})$ is invertible over U if and only if $\mathbf{g}|_{\hbar=0}$ is invertible. We denote the inverse matrix of \mathbf{g} by (\mathbf{g}^{ij}) . In this case, X reduces to an embedded

space with metric $\mathbf{g}|_{\hbar=0}$ in the commutative limit with $\theta = 0$. Therefore, we call X a *noncommutative space* embedded in \mathcal{A}^m in analogy to the classical case.

Let $E_i = \partial_i X$ for $i = 1, 2, \dots, n$. Then $(E_i)^t = \begin{pmatrix} \partial_i X^1 \\ \partial_i X^2 \\ \vdots \\ \partial_i X^m \end{pmatrix}$. The matrix

$$e = \eta(E_i)^t * \mathbf{g}^{ij} * E_j$$

is an idempotent and satisfies $E_i * e = E_i$ and $e * \eta(E_i)^t = \eta(E_i)^t$ for all i . The left (resp. right) projective \mathcal{A} -module $\mathcal{M} = {}_l\mathcal{A}^m * e$ (resp. $\mathcal{M} = e * \mathcal{A}_r^m$) associated to e is the quantised left (resp. right) tangent bundle of the embedded noncommutative space. It is easy to show that the metric defined by the \mathcal{A} -bilinear map (2.3) agrees with (2.5) in the present case.

We may cast the formulation of the embedded noncommutative space into a more familiar form. The connection is now given by

$$\nabla_i E_j = \Gamma_{ij}^k * E_k,$$

where Γ_{ij}^k can be explicitly described in the following way. Let $\Gamma_{ijl} = \Gamma_{ij}^k * \mathbf{g}_{kl}$. We have

$$\Gamma_{ijl} = {}_c\Gamma_{ijl} + \Upsilon_{ilj} + \Upsilon_{jil} - \Upsilon_{lji},$$

with

$${}_c\Gamma_{ijl} = \frac{1}{2} (\partial_i \mathbf{g}_{jl} + \partial_j \mathbf{g}_{li} - \partial_l \mathbf{g}_{ji}), \quad \Upsilon_{ijl} = \frac{1}{2} (\partial_i E_j * \eta(E_l)^t - E_l * \eta \partial_i (E_j)^t),$$

where the object Υ_{ijl} is called the noncommutative torsion in [6]. The curvatures are given by

$$\begin{aligned} R_{kij}^l &= -\partial_j \Gamma_{ik}^l - \Gamma_{ik}^p * \Gamma_{jp}^l + \partial_i \Gamma_{jk}^l + \Gamma_{jk}^p * \Gamma_{ip}^l, \\ R_j^i &= \mathbf{g}^{ik} * R_{kpj}^p, \quad \Theta_p^l = \mathbf{g}^{ik} * R_{kpi}^l. \end{aligned}$$

It was shown in [6, 7] that the two noncommutative scalar curvatures R and Θ coincide in the present case.

3. EXACT SOLUTIONS OF NONCOMMUTATIVE EINSTEIN FIELD EQUATIONS

We shall now construct a class of exact solutions of the noncommutative vacuum Einstein field equations. The solutions are quantum deformed analogues of plane-fronted gravitational waves [17, 18, 19, 20].

Let (θ_{ij}) be an arbitrary constant skew symmetric 4×4 matrix, and endow the space of functions of the variables (x, y, u, v) with the Moyal product defined with respect to (θ_{ij}) . We denote the resulting noncommutative algebra by \mathcal{A} .

Now we consider a noncommutative space X embedded in \mathcal{A}^6 by a map of the form

$$(3.1) \quad X = \left(x, y, \frac{Hu + u + v}{\sqrt{2}}, \frac{H - \frac{u^2}{2}}{\sqrt{2}}, \frac{Hu - u + v}{\sqrt{2}}, \frac{H + \frac{u^2}{2}}{\sqrt{2}} \right),$$

where, needless to say, the component functions are elements of \mathcal{A} .

Let us take $\eta = \text{diag}(1, 1, 1, 1, -1, -1)$, and construct the noncommutative metric \mathbf{g} by using the formula (2.5) for this embedded noncommutative space. A very lengthy calculation yields the following result:

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & -\bar{h}(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) & 0 \\ 0 & 1 & -\bar{h}(\theta_{yu}H_{yy} + \theta_{xu}H_{xy}) & 0 \\ \bar{h}(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) & \bar{h}(\theta_{yu}H_{yy} + \theta_{xu}H_{xy}) & 2H & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is useful to note that in the classical limit with all $\theta_{ij} = 0$, the metric has

Minkowski signature. In fact it reduces to the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2H & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, which

diagonalises to $\text{diag}(1, 1, H + \sqrt{1 + H^2}, H - \sqrt{1 + H^2})$. Further tedious computations produce the following inverse metric:

$$\mathbf{g}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \bar{h}(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) \\ 0 & 1 & 0 & \bar{h}(\theta_{yu}H_{yy} + \theta_{xu}H_{xy}) \\ 0 & 0 & 0 & 1 \\ -\bar{h}(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) & -\bar{h}(\theta_{yu}H_{yy} + \theta_{xu}H_{xy}) & 1 & g^{44} \end{pmatrix}$$

with

$$g^{44} = -\bar{h}^2(\theta_{yu}H_{yy} + \theta_{xu}H_{xy}) * (\theta_{yu}H_{yy} + \theta_{xu}H_{xy}) \\ - \bar{h}^2(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) * (\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) - 2H.$$

Using these formulae we can compute Γ_{ijk} and Γ_{ij}^k , the nonzero components of which are given below:

$$\begin{aligned} \Gamma_{113} &= -\bar{h}(\theta_{yu}H_{xxy} + \theta_{xu}H_{xxx}), \\ \Gamma_{123} &= \Gamma_{213} = -\bar{h}(\theta_{yu}H_{xyy} + \theta_{xu}H_{xxy}), \\ \Gamma_{133} &= \Gamma_{313} = H_x - \bar{h}(\theta_{yu}H_{xyu} + \theta_{xu}H_{xxu}), \\ \Gamma_{223} &= -\bar{h}(\theta_{yu}H_{yyy} + \theta_{xu}H_{xyy}), \\ \Gamma_{233} &= \Gamma_{323} = H_y - \bar{h}(\theta_{yu}H_{yyu} + \theta_{xu}H_{xyu}), \\ \Gamma_{331} &= -H_x, \quad \Gamma_{332} = -H_y, \\ \Gamma_{333} &= H_u - \bar{h}(\theta_{yu}H_{yuu} + \theta_{xu}H_{xuu}); \\ \Gamma_{11}^4 &= -\bar{h}(\theta_{yu}H_{xxy} + \theta_{xu}H_{xxx}), \\ \Gamma_{12}^4 &= \Gamma_{21}^4 = -\bar{h}(\theta_{yu}H_{xyy} + \theta_{xu}H_{xxy}), \\ \Gamma_{13}^4 &= \Gamma_{31}^4 = H_x - \bar{h}(\theta_{yu}H_{xyu} + \theta_{xu}H_{xxu}), \\ \Gamma_{22}^4 &= -\bar{h}(\theta_{yu}H_{yyy} + \theta_{xu}H_{xyy}), \\ \Gamma_{23}^4 &= \Gamma_{32}^4 = H_y - \bar{h}(\theta_{yu}H_{yyu} + \theta_{xu}H_{xyu}), \\ \Gamma_{33}^1 &= -H_x, \quad \Gamma_{33}^2 = -H_y, \\ \Gamma_{33}^4 &= -H_x * \bar{h}(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) - H_y * \bar{h}(\theta_{yu}H_{yy} + \theta_{xu}H_{xy}) \\ &\quad + H_u - \bar{h}(\theta_{yu}H_{yuu} + \theta_{xu}H_{xuu}). \end{aligned}$$

Remarkably, explicit formulae for curvatures can also be obtained, even though the noncommutativity of the $*$ -product complicates the computations enormously. As an example, we give the computation of R_{3313} here:

$$\begin{aligned}
R_{3313} &= \frac{\partial \Gamma_{33}^p}{\partial x} * \mathbf{g}_{p3} - \frac{\partial \Gamma_{13}^p}{\partial u} * \mathbf{g}_{p3} + \Gamma_{33}^p * \Gamma_{1p3} - \Gamma_{13}^p * \Gamma_{3p3} \\
&= \frac{\partial \Gamma_{33}^1}{\partial x} * \mathbf{g}_{13} + \frac{\partial \Gamma_{33}^2}{\partial x} * \mathbf{g}_{23} + \frac{\partial \Gamma_{33}^4}{\partial x} * \mathbf{g}_{43} - \frac{\partial \Gamma_{13}^4}{\partial u} * \mathbf{g}_{43} \\
&\quad + \Gamma_{33}^1 * \Gamma_{113} + \Gamma_{33}^2 * \Gamma_{123} \\
&= -H_{xx} * (-\bar{h}(\theta - yuH_{xy} + \theta_{xu}H_{xx})) \\
&\quad - H_{xy} * (-\bar{h}(\theta_{yu}H_{yy} + \theta_{xu}H_{xy})) \\
&\quad + \frac{\partial \Gamma_{33}^4}{\partial x} - H_{xu} + \bar{h}(\theta_{yu}H_{xyuu} + \theta_{xu}H_{xxuu}) \\
&\quad + (-H_x) * (-\bar{h}(\theta_{yu}H_{xxy} + \theta_{xu}H_{xxx})) \\
&\quad + (-H_y) * (-\bar{h}(\theta_{yu}H_{xyy} + \theta_{xu}H_{xxy})) = 0.
\end{aligned}$$

The other components of the curvature can be obtained in the same way. We have

$$\begin{aligned}
R_{1313} &= -R_{1331} = -H_{xx}, & R_{1323} &= -R_{1332} = -H_{xy}, \\
R_{2313} &= -R_{2331} = -H_{xy}, & R_{2323} &= -R_{2332} = -H_{yy}, \\
R_{3113} &= -R_{3131} = H_{xx}, & R_{3123} &= -R_{3132} = H_{xy}, \\
R_{3213} &= -R_{3231} = H_{xy}, & R_{3223} &= -R_{3232} = H_{yy}, \\
R_{3331} &= -R_{3313} = 0, & R_{3332} &= -R_{3323} = 0.
\end{aligned}$$

Thus the nonzero components of R_{ijk}^l are

$$\begin{aligned}
R_{113}^4 &= -R_{131}^4 = H_{xx}, & R_{123}^4 &= -R_{132}^4 = H_{xy}, \\
R_{213}^4 &= -R_{231}^4 = H_{xy}, & R_{223}^4 &= -R_{232}^4 = H_{yy}, \\
R_{313}^1 &= -R_{331}^1 = -H_{xx}, & R_{313}^2 &= -R_{331}^2 = -H_{xy}, \\
R_{323}^1 &= -R_{332}^1 = -H_{xy}, & R_{323}^2 &= -R_{332}^2 = -H_{yy}, \\
R_{313}^4 &= -R_{331}^4 = -H_{xx} * \bar{h}(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) \\
&\quad - H_{xy} * \bar{h}(\theta_{yu}H_{yy} + \theta_{xu}H_{xy}), \\
R_{323}^4 &= -R_{332}^4 = -H_{xy} * \bar{h}(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) \\
&\quad - H_{yy} * \bar{h}(\theta_{yu}H_{yy} + \theta_{xu}H_{xy}).
\end{aligned}$$

From these formulae, we obtain the nonzero components of the Ricci curvature:

$$(3.2) \quad R_3^4 = \Theta_3^4 = -H_{xx} - H_{yy}.$$

Thus the noncommutative vacuum Einstein field equations (2.4) are satisfied if and only if the following equation holds:

$$(3.3) \quad H_{xx} + H_{yy} = 0.$$

Solutions of this linear equation for H exist in abundance. Each solution leads to an exact solution of the noncommutative vacuum Einstein field equations. If we set θ to zero, we recover from such a solution the plane-fronted gravitational wave [17, 18, 19, 20] in classical general relativity. Thus we shall call such a solution of (2.4) a *plane-fronted noncommutative gravitational wave*.

It is clear from (3.2) that plane-fronted noncommutative gravitational waves satisfy the additivity property. Explicitly, if the noncommutative metrics of

$$X^{(i)} = \left(x, y, \frac{H_i u + u + v}{\sqrt{2}}, \frac{H_i - \frac{u^2}{2}}{\sqrt{2}}, \frac{H_i u - u + v}{\sqrt{2}}, \frac{H_i + \frac{u^2}{2}}{\sqrt{2}} \right), \quad i = 1, 2,$$

are plane-fronted noncommutative gravitational waves, we let $H = H_1 + H_2$, and set

$$X = \left(x, y, \frac{Hu + u + v}{\sqrt{2}}, \frac{H - \frac{u^2}{2}}{\sqrt{2}}, \frac{Hu - u + v}{\sqrt{2}}, \frac{H + \frac{u^2}{2}}{\sqrt{2}} \right).$$

Then the noncommutative metric of X is also a plane-fronted noncommutative gravitational wave. This is a rather nontrivial fact since the noncommutative Einstein field equations are highly nonlinear in \mathbf{g} , and it is extremely rare to have this additivity property.

At this point, it is appropriate to point out that the embedding (3.1) is only used as a device for constructing the metric and the connection, from which the curvatures are derived. However, we should observe the power of embeddings in solving the noncommutative Einstein field equations. Without using the embedding (3.1), it would be very difficult to come up with elegant solutions like what we have obtained here.

4. DISCUSSIONS

Working within the framework of the noncommutative Riemannian geometry of [6], we have obtained in this paper exact solutions of the quantum noncommutative vacuum Einstein field equations, which are noncommutative analogues of the plane-fronted gravitational waves in classical general relativity [17, 18, 19, 20].

In the classical setting, the plane-fronted gravitational waves model spacetimes moving at the speed of light and radiating energy. Furthermore, Penrose [25] observed that near a null geodesic, every spacetime can be blown up so that the given null geodesic becomes the covariantly constant null geodesic congruence of a plane wave. We expect the plane-fronted noncommutative gravitational waves to play a similar role. It will be very interesting to investigate the physical applications of these solutions.

It is quite striking that the quantum noncommutative Einstein field equations [6], complicated as they are, admit explicit exact solutions as simple as the ones constructed here. This indicates the promise of the theory of noncommutative Riemannian geometry proposed in [6]. We hope that the theory will develop into a coherent framework for studying the structure of spacetime at the Planck scale.

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INSTITUTE OF MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA

E-mail address: wangding@amss.ac.cn

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, SYDNEY, NSW 2006, AUSTRALIA

E-mail address: rzhang@maths.usyd.edu.au

INSTITUTE OF MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA

E-mail address: xzhang@amss.ac.cn